Improving Coupling of Multiscale Computational Models within the Adaptive Hydraulics Framework

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Overview

Introduction

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Introduction

Adaptive Hydraulics (AdH): (US Army Corps of Engineers)

- 2D and 3D shallow water (SW) equations
- 3D Navier-Stokes equations
- Barotropic/baroclinic transport of constituents (salinity, temperature, sediments, etc.)
- Spatiotemporal adaptivity

Objective

"To algebraically couple 2D and 3D shallow water models in AdH, in a conservative manner"

Solution method

Spatial discretization of PDE’s to generate nonlinear ODE’s in time

- Streamline upwind Petrov-Galerkin (SUPG) method

Temporal discretization of nonlinear ODE’s to generate a system of nonlinear equations

- Up to second order implicit finite difference method

Nonlinear equations solved using Newton-Raphson iterative method

AdH: 2D SW models

Equations after applying SUPG

- Depth integrated continuity equation (1)

$$\sum_e \left[\underbrace{\int_{\Omega_e^{2D}} \left[\phi_i \frac{\partial h}{\partial t} - \nabla^{2D} \phi_i \cdot (\bar{\boldsymbol{v}} h) \right] d\Omega_e^{2D}}_{Interior\ terms\ (IBP)} + \oint_{\partial\Omega_e^{1D}} \underbrace{[\phi_i (\bar{\boldsymbol{v}} h) \cdot \boldsymbol{n}] d\partial\Omega_e^{1D}}_{Boundary\ terms\ (mass\ flux)} + \underbrace{\bar{P}_e^c}_{SUPG\ terms} \right] = 0 \tag{1}$$

- Depth integrated horizontal momentum equations (2) and (3)

$$\sum_e \left[\underbrace{\int_{\Omega_e^{2D}} \left[\phi_i \frac{\partial (\bar{u} h)}{\partial t} - \frac{\partial \phi_i}{\partial x} \left(\bar{u}\bar{u}h + \frac{gh^2}{2} - \frac{h\sigma_{xx}}{\rho} \right) - \frac{\partial \phi_i}{\partial y} \left(\bar{u}\bar{v}h - \frac{h\sigma_{xy}}{\rho} \right) \right] d\Omega_e^{2D}}_{Interior\ terms\ (IBP)} + \oint_{\partial\Omega_e^{1D}} \underbrace{\left[\phi_i n_x \left(\bar{u}\bar{u}h + \frac{gh^2}{2} - \frac{h\sigma_{xx}}{\rho} \right) + \phi_i n_y \left(\bar{u}\bar{v}h - \frac{h\sigma_{xy}}{\rho} \right) \right] d\partial\Omega_e^{1D}}_{Boundary\ terms\ (momentum\ flux)} + \underbrace{\bar{P}_e^{mx}}_{SUPG\ terms} \right] = 0 \tag{2}$$

$$\sum_e \left[\underbrace{\oint_{\Omega_e^{2D}} \left[\phi_i \frac{\partial (\bar{v} h)}{\partial t} - \frac{\partial \phi_i}{\partial x} \left(\bar{v}\bar{u}h - \frac{h\sigma_{yx}}{\rho} \right) - \frac{\partial \phi_i}{\partial y} \left(\bar{v}\bar{v}h + \frac{gh^2}{2} - \frac{h\sigma_{yy}}{\rho} \right) \right] d\Omega_e^{2D}}_{Interior\ terms\ (IBP)} + \oint_{\partial\Omega_e^{1D}} \underbrace{\left[\phi_i n_x \left(\bar{v}\bar{u}h - \frac{h\sigma_{yx}}{\rho} \right) + \phi_i n_y \left(\bar{v}\bar{v}h + \frac{gh^2}{2} - \frac{h\sigma_{yy}}{\rho} \right) \right] d\partial\Omega_e^{1D}}_{Boundary\ terms\ (momentum\ flux)} + \underbrace{\bar{P}_e^{my}}_{SUPG\ terms} \right] = 0 \tag{3}$$

AdH: 3D SW models

Equations after applying SUPG

- Depth summed continuity equation (4)

$$\sum_{i\in\mathcal{C}(I)}\sum_{e}\left[\int_{\Omega_e^{3D}}\underbrace{[-\nabla\phi_i\cdot\boldsymbol{v}]d\Omega_e^{3D}}_{Interior\ terms\ (IBP)}\right]+\sum_{e}\left[\int_{\partial\Omega_{e,s}^{2D}}\underbrace{\left[\phi_i\frac{\partial\eta}{\partial t}n_z\right]d\partial\Omega_{e,s}^{2D}}_{Surface\ mass\ flux}+\int_{\partial\Omega_{e,b}^{2D}}\underbrace{\left[\phi_i\frac{\partial b}{\partial t}n_z\right]d\partial\Omega_{e,b}^{2D}}_{Bed\ mass\ flux}\right]+\sum_{i\in\mathcal{C}(I)}\sum_{e}\int_{\partial\Omega_{e,v}^{2D}}\underbrace{[\phi_i\boldsymbol{v}\cdot\boldsymbol{n}]d\partial\Omega_{e,v}^{2D}}_{Vertical\ boundary\ mass\ flux}+\sum_{i\in\mathcal{C}(I)}\sum_{e}\underbrace{P_e^C}_{SUPG\ terms}=0 \tag{4}$$

- Horizontal momentum equations (5) and (6)

$$\sum_{e}\left[\frac{\partial}{\partial t}\int_{\Omega_e^{3D}}\underbrace{[\phi_i u]d\Omega_e^{3D}}_{Interior\ terms\ (IBP)}+\int_{\Omega_e^{3D}}\underbrace{\left[-\nabla\phi_i\cdot(\boldsymbol{v}_r u)-\phi_i f v+\frac{\partial\phi_i}{\partial x}\frac{P}{\rho_0}+\nabla\phi_i\cdot\boldsymbol{\tau}_x\right]d\Omega_e^{3D}}_{Interior\ terms\ (IBP)}+\int_{\partial\Omega_{e,v}^{2D}}\underbrace{\left[\phi_i\boldsymbol{n}\cdot(\boldsymbol{v}_r u)+n_x\frac{P}{\rho_0}\right]d\partial\Omega_{e,v}^{2D}}_{Vertical\ boundary\ momentum\ flux}-\int_{\partial\Omega_{e,w}^{2D}}\underbrace{[\phi_i(\boldsymbol{\tau}_x\cdot\boldsymbol{n})]d\partial\Omega_{e,w}^{2D}}_{Side\ wall\ stress}+\underbrace{P_e^{mx}}_{SUPG\ terms}\right]=0 \tag{5}$$

$$\sum_{e}\left[\frac{\partial}{\partial t}\int_{\Omega_e^{3D}}\underbrace{[\phi_i v]d\Omega_e^{3D}}_{Interior\ terms\ (IBP)}+\int_{\Omega_e^{3D}}\underbrace{\left[-\nabla\phi_i\cdot(\boldsymbol{v}_r v)+\phi_i f u+\frac{\partial\phi_i}{\partial y}\frac{P}{\rho_0}+\nabla\phi_i\cdot\boldsymbol{\tau}_y\right]d\Omega_e^{3D}}_{Interior\ terms\ (IBP)}+\int_{\partial\Omega_{e,v}^{2D}}\underbrace{\left[\phi_i\boldsymbol{n}\cdot(\boldsymbol{v}_r v)+n_y\frac{P}{\rho_0}\right]d\partial\Omega_{e,v}^{2D}}_{Vertical\ boundary\ momentum\ flux}-\int_{\partial\Omega_{e,w}^{2D}}\underbrace{[\phi_i(\boldsymbol{\tau}_y\cdot\boldsymbol{n})]d\partial\Omega_{e,w}^{2D}}_{Side\ wall\ stress}+\underbrace{P_e^{my}}_{SUPG\ terms}\right]=0 \tag{6}$$

AdH: Final equations

2D SW models:

$$\boldsymbol{r}_{2D}^{1} = \begin{cases} r_{1,2D}^{mx}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0 \\ r_{1,2D}^{my}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0 \\ r_{1,2D}^{c}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0 \end{cases}$$

$$\boldsymbol{r}_{2D}^{2} = \begin{cases} r_{2,2D}^{mx}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0 \\ r_{2,2D}^{my}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0 \\ r_{2,2D}^{c}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0 \end{cases}$$

$$\vdots$$

$$r_{N,2D}^{c}\big(\boldsymbol{s}_{2D}(t^{n+1})\big) = 0$$

where

$$\boldsymbol{s}_{2D} = \{\bar{u}_1, \bar{v}_1, h_1, \dots, \bar{u}_N, \bar{v}_N, h_N\}^T$$

3D SW models:

$$\boldsymbol{r}_{3D}^{1} = \begin{cases} r_{1,3D}^{mx}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0 \\ r_{1,3D}^{my}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0 \\ r_{1,3D}^{c}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0 \end{cases}$$

$$\boldsymbol{r}_{3D}^{2} = \begin{cases} r_{2,3D}^{mx}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0 \\ r_{2,3D}^{my}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0 \\ r_{2,3D}^{c}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0 \end{cases} \qquad (7)$$

$$\vdots$$

$$r_{N,3D}^{c}\big(\boldsymbol{s}_{3D}(t^{n+1})\big) = 0$$

where

$$\boldsymbol{s}_{3D} = \{u_1, v_1, d_1, \dots, u_N, v_N, d_N\}^T$$

AdH: Final equations

Nonlinear equations in vector form for both 2D and 3D models can be written as (8)

$$\boldsymbol{R}(\boldsymbol{s}_{n+1}) = \boldsymbol{0} \tag{8}$$

Newton-Raphson iterations (9) are set up to solve (8)

$$\begin{aligned} \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{s}_{n+1}}\right)^{(i)} \Delta \boldsymbol{s}_{n+1}^{(i+1)} &= -\boldsymbol{R}\left(\boldsymbol{s}_{n+1}^{(i)}\right) \\ \boldsymbol{s}_{n+1}^{(i+1)} &= \boldsymbol{s}_{n+1}^{(i)} + \Delta \boldsymbol{s}_{n+1}^{(i+1)} \end{aligned} \tag{9}$$

AdH: Final equations

Newton-Raphson iterations (9) for 2D SW models, for example, look like (10)

$$
\begin{bmatrix}
\frac{\partial r_1^{mx}}{\partial \bar{u}_1} & \frac{\partial r_1^{mx}}{\partial \bar{v}_1} & \frac{\partial r_1^{mx}}{\partial h_1} & \frac{\partial r_1^{mx}}{\partial \bar{u}_2} & \frac{\partial r_1^{mx}}{\partial \bar{v}_2} & \frac{\partial r_1^{mx}}{\partial h_2} & \cdots & \frac{\partial r_1^{mx}}{\partial h_N} \\
\frac{\partial r_1^{my}}{\partial \bar{u}_1} & \frac{\partial r_1^{my}}{\partial \bar{v}_1} & \frac{\partial r_1^{my}}{\partial h_1} & \frac{\partial r_1^{my}}{\partial \bar{u}_2} & \frac{\partial r_1^{my}}{\partial \bar{v}_2} & & \cdots & \frac{\partial r_1^{my}}{\partial h_N} \\
\frac{\partial r_1^{c}}{\partial \bar{u}_1} & \frac{\partial r_1^{c}}{\partial \bar{v}_1} & \frac{\partial r_1^{c}}{\partial h_1} & \frac{\partial r_1^{c}}{\partial \bar{u}_2} & & & \cdots & \frac{\partial r_1^{c}}{\partial h_N} \\
\frac{\partial r_2^{mx}}{\partial \bar{u}_1} & \frac{\partial r_2^{mx}}{\partial \bar{v}_1} & & & & & \cdots & \frac{\partial r_2^{mx}}{\partial h_N} \\
\frac{\partial r_2^{my}}{\partial \bar{u}_1} & & & & & & \cdots & \frac{\partial r_2^{my}}{\partial h_N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial r_N^{c}}{\partial \bar{u}_1} & \frac{\partial r_N^{c}}{\partial \bar{v}_1} & \frac{\partial r_N^{c}}{\partial h_1} & \frac{\partial r_N^{c}}{\partial \bar{u}_2} & \frac{\partial r_N^{c}}{\partial \bar{v}_2} & \frac{\partial r_N^{c}}{\partial h_2} & \cdots & \frac{\partial r_N^{c}}{\partial h_N}
\end{bmatrix}^{(i)}
\begin{Bmatrix}
\Delta\bar{u}_1 \\ \Delta\bar{v}_1 \\ \Delta h_1 \\ \Delta\bar{u}_2 \\ \Delta\bar{v}_2 \\ \vdots \\ \Delta h_N
\end{Bmatrix}^{(i+1)}
= -
\begin{Bmatrix}
r_1^{mx}\left(\boldsymbol{s}_{n+1}^{(i)}\right) \\
r_1^{my}\left(\boldsymbol{s}_{n+1}^{(i)}\right) \\
r_1^{c}\left(\boldsymbol{s}_{n+1}^{(i)}\right) \\
r_2^{mx}\left(\boldsymbol{s}_{n+1}^{(i)}\right) \\
r_2^{my}\left(\boldsymbol{s}_{n+1}^{(i)}\right) \\
\vdots \\
r_N^{c}\left(\boldsymbol{s}_{n+1}^{(i)}\right)
\end{Bmatrix}
\qquad (10)
$$

(Iteration index 'i' and time step number 'n' dropped hereafter)

Notation

$\mathcal{N}^{2D}$	Set of all nodes in the 2D model
$\mathcal{I}^{2D}$	Set of all nodes in the 2D model that lie on the 2D-3D interface
$\boldsymbol{R}_{2D}$	Global residual vector of the 2D domain
$\boldsymbol{r}_{2D}^{i} = \left\{r_{i,2D}^{mx}, r_{i,2D}^{my}, r_{i,2D}^{c}\right\}^T$	Nonlinear residual vector at node i of the 2D domain
$\boldsymbol{s}_{2D}^{i} = \{\bar{u}_i, \bar{v}_i, h_i\}^T$	Solution vector at node i of the 2D domain
$\mathcal{N}^{3D}$	Set of all nodes in the 3D model
$\mathcal{I}^{3D}$	Set of all nodes in the 3D model that lie on the 2D-3D interface
$\boldsymbol{R}_{3D}$	Global residual vector of the 3D domain
$\boldsymbol{r}_{3D}^{i} = \left\{r_{i,3D}^{mx}, r_{i,3D}^{my}, r_{i,3D}^{c}\right\}^T$	Nonlinear residual vector at node i of the 3D domain
$\boldsymbol{s}_{3D}^{i} = \{u_i, v_i, d_i\}^T$	Solution vector at node i of the 3D domain

Notation

$\boldsymbol{R} = \left\{ \boldsymbol{R}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}, \boldsymbol{R}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}, \boldsymbol{R}^{\mathcal{J}^{2D}}, \boldsymbol{R}^{\mathcal{J}^{3D}} \right\}^T$	(Rearranged) global residual vector of the coupled domain
$\boldsymbol{s} = \left\{ \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}, \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}, \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}, \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}} \right\}^T$	(Rearranged) global solution vector of the coupled domain
$\boldsymbol{r}_i = \left\{ r_i^{mx}, r_i^{my}, r_i^{c} \right\}^T$	New residual vector at node i of the coupled domain
$\boldsymbol{s}^i = \begin{cases} \boldsymbol{s}_{2D}^i, & if\ i \in \mathcal{N}^{2D} \\ \boldsymbol{s}_{3D}^i, & if\ i \in \mathcal{N}^{3D} \end{cases}$	Solution vector at node i of the coupled domain
$\boldsymbol{c}_i$	Linear constraint applied at node i of the 3D interface
$\boldsymbol{C} = \{\boldsymbol{c}_i\}^T$	Constraint vector for all nodes on the 3D interface
$\mathcal{K}$	A node on the 2D model interface
$\mathcal{C}(\mathcal{K})$	The column (set) of 3D model interface nodes that $\mathcal{K}$ is coupled to

Example

Primary set definitions:

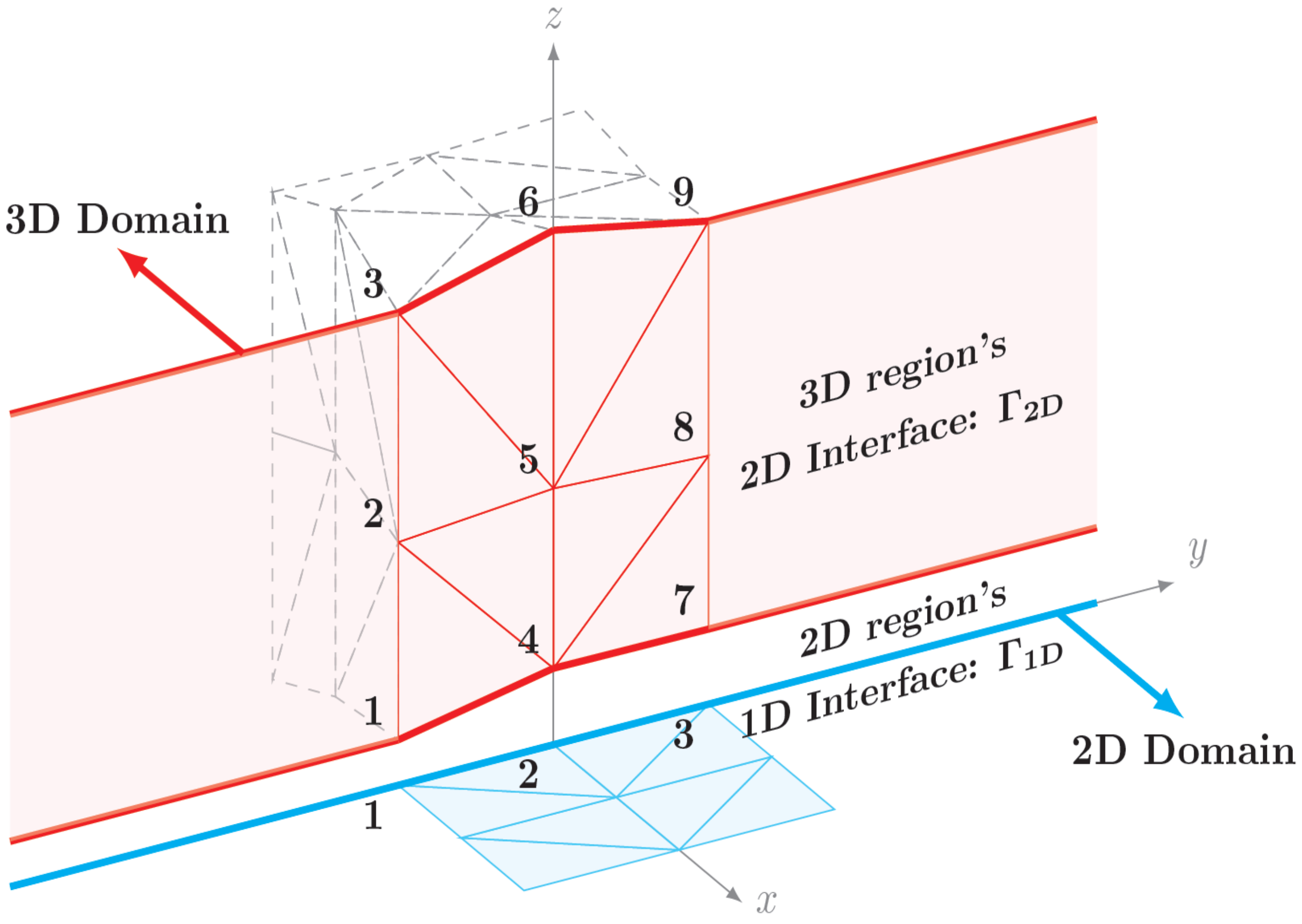
$\mathcal{N}^{2D} = \{All\ nodes\ in\ 2D\ model\}$
$\mathcal{N}^{3D} = \{All\ nodes\ in\ 3D\ model\}$

Separate Interface Nodes:

$\mathcal{I}^{2D} = \{1_{2D}, 2_{2D}, 3_{2D}\}$
$\mathcal{I}^{3D} = \{1_{3D}, 2_{3D}, 3_{3D}, 4, \ldots, 9\}$

Node Columns:

$\mathcal{C}(\mathcal{K} = 1_{2D}) = \{1_{3D}, 2_{3D}, 3_{3D}\}$
$\mathcal{C}(\mathcal{K} = 2_{2D}) = \{4, 5, 6\}$
$\mathcal{C}(\mathcal{K} = 3_{2D}) = \{7, 8, 9\}$



Coupling: Old system

Newton iterations (9) for the combined, reordered (non-coupled) 2D-3D system are given by (11)

$$\begin{bmatrix} \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\ [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\ \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\ [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} \\ \Delta \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}} \end{Bmatrix} = - \begin{Bmatrix} \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}(\boldsymbol{s}_{2D}) \\ \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}(\boldsymbol{s}_{3D}) \\ \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}}(\boldsymbol{s}_{2D}) \\ \boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}(\boldsymbol{s}_{3D}) \end{Bmatrix} \quad (11)$$

Coupling: New residuals

Define the 'new,' coupled residuals using (12)

$$\boldsymbol{R} \ni \boldsymbol{r}^j = \begin{cases} \boldsymbol{r}_{2D}^j & \forall j \in \mathcal{N}^{2D} - \mathcal{J}^{2D} \\ \boldsymbol{r}_{3D}^j & \forall j \in \mathcal{N}^{3D} - \mathcal{J}^{3D} \\ \boldsymbol{r}_{2D}^j + \sum_{i \in \mathcal{C}(j)} \boldsymbol{r}_{3D}^i & \forall j \in \mathcal{J}^{2D} \\ \boldsymbol{s}_{3D}^j - \boldsymbol{s}_{2D}^{\mathcal{K}} & \forall j \in \mathcal{J}^{3D}, where\ \exists!\, \mathcal{K} \in \mathcal{J}^{2D} : j \in \mathcal{C}(\mathcal{K}) \end{cases} \tag{12}$$

In vector form, (12) can be informally rewritten as (13)

(Note: Σ^* is not a function, just notation)

$$\begin{aligned} \boldsymbol{R}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} &= \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} \\ \boldsymbol{R}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} &= \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} \\ \boldsymbol{R}^{\mathcal{J}^{2D}} &= \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}} + \Sigma^*\left(\boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}\right) \\ \boldsymbol{R}^{\mathcal{J}^{3D}} &= \boldsymbol{C} \end{aligned} \tag{13}$$

Coupling: New Jacobian

Derivatives for new residuals (12) at the 2D interface nodes are given by (14)

$$\frac{\partial \boldsymbol{r}^{\mathcal{K}}}{\partial \boldsymbol{s}} = \frac{\partial \boldsymbol{r}_{2D}^{\mathcal{K}}}{\partial \boldsymbol{s}} + \sum_{j \in \mathcal{C}(\mathcal{K}) \subset \mathcal{J}^{3D}} \frac{\partial \boldsymbol{r}_{3D}^{j}}{\partial \boldsymbol{s}} \qquad \forall \mathcal{K} \in \mathcal{J}^{2D} \tag{14}$$

Eq. (14) can be reinterpreted as (15) for programming purposes

$$\begin{matrix} Block\ row\ (\mathcal{K}) \\ (New\ Jacobian) \end{matrix} = \begin{matrix} Block\ row\ (\mathcal{K}) \\ (Old\ Jacobian) \end{matrix} + \sum_{j \in \mathcal{C}(\mathcal{K})} \begin{matrix} Block\ row\ (j) \\ (Old\ Jacobian) \end{matrix} \quad \forall \mathcal{K} \in \mathcal{J}^{2D} \tag{15}$$

Coupling: New Jacobian

Derivatives for new residuals (12) at the 3D interface nodes are given by (16)

$$\frac{\partial \boldsymbol{r}^j}{\partial \boldsymbol{s}^i} = \frac{\partial \boldsymbol{s}_{3D}^j}{\partial \boldsymbol{s}^i} - \frac{\partial \boldsymbol{s}_{2D}^{\mathcal{K}}}{\partial \boldsymbol{s}^i} = \begin{cases} +[I], & if\ i = j \\ -[I], & if\ i = \mathcal{K} \\ [0], & otherwise \end{cases} \qquad \begin{matrix} \forall j \in \mathcal{J}^{3D}, where \\ \exists! \, \mathcal{K} \in \mathcal{J}^{2D}: j \in \mathcal{C}(\mathcal{K}) \end{matrix} \tag{16}$$

Eq. (16) can be reinterpreted as (17) for programming purposes

$$\left.\begin{matrix} New\ Jacobian\ Block[j][j] = +[I] \\ New\ Jacobian\ Block[j][\mathcal{K}] = -[I] \\ New\ Jacobian\ Block[j][i] = [0] \end{matrix}\right\} \qquad \begin{matrix} \forall j \in \mathcal{J}^{3D}, where \\ \exists! \, \mathcal{K} \in \mathcal{J}^{2D}: j \in \mathcal{C}(\mathcal{K}) \\ and\ i \neq j, \mathcal{K} \end{matrix} \tag{17}$$

Coupling: New system

Use the new residuals (12), and derivatives (14) and (16), to modify the non-coupled system (11), to get the coupled system (18)

$$\begin{bmatrix} \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\ [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\ \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & \frac{\partial \Sigma^{*}\left(\boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}\right)}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & \frac{\partial \Sigma^{*}\left(\boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}\right)}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\ [0] & [0] & \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} \\ \Delta \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}} \end{Bmatrix} = - \begin{Bmatrix} \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}(\boldsymbol{s}_{2D}) \\ \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}(\boldsymbol{s}_{3D}) \\ \boldsymbol{R}_{2D}^{\mathcal{J}^{2D}}(\boldsymbol{s}_{2D}) + \Sigma^{*}\left(\boldsymbol{R}_{3D}^{\mathcal{J}^{3D}}(\boldsymbol{s}_{3D})\right) \\ \boldsymbol{C}\left(\boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}, \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}\right) \end{Bmatrix} \quad (18)$$

Solve (20) and update the solution vector

Check if nonlinear equations (8) are satisfied within user-defined tolerance

- If YES, then increment time step
- If NO, perform the next Newton-Raphson iteration

Example

Primary set definitions:

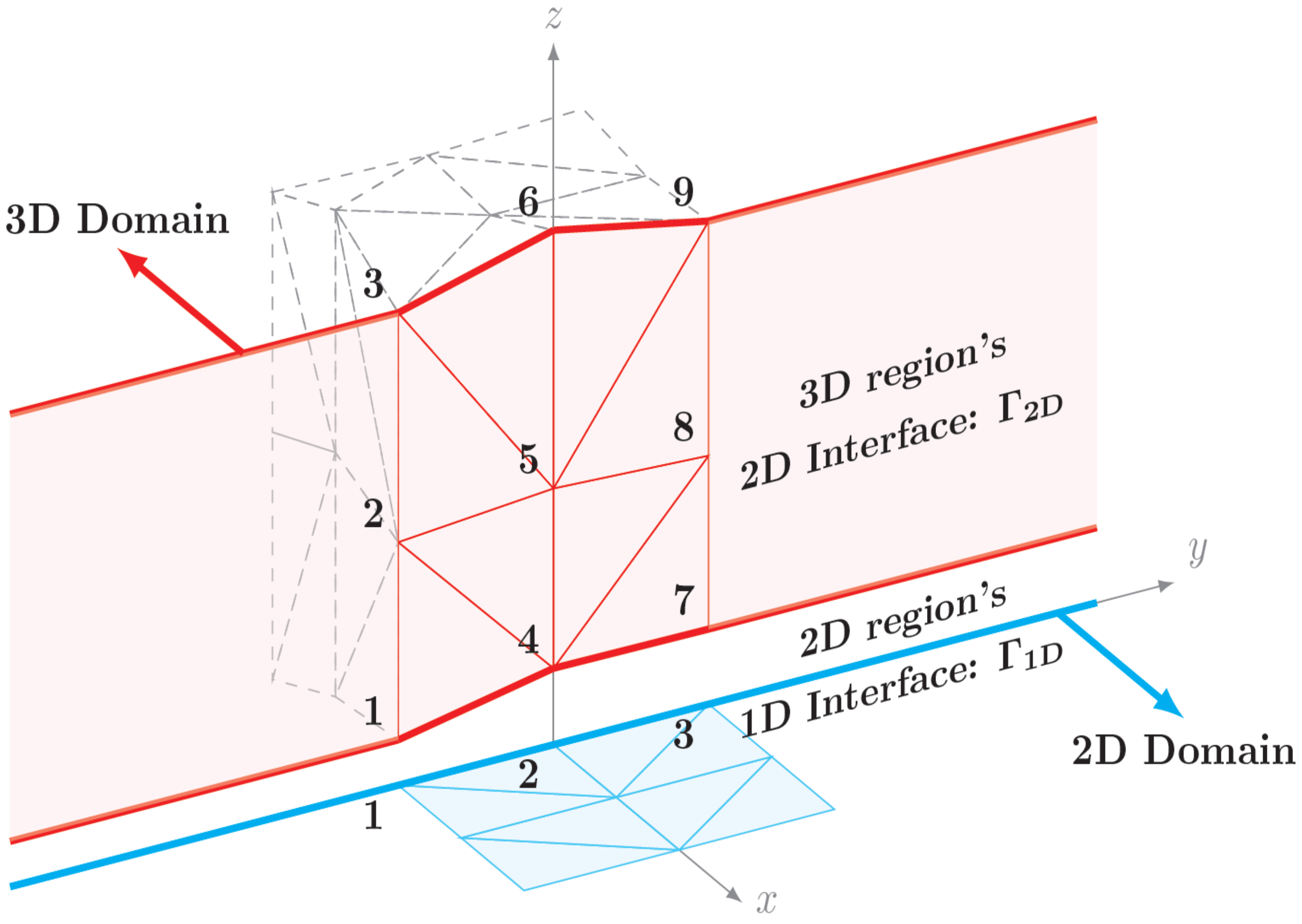
$\mathcal{N}^{2D} = \{All\ nodes\ in\ 2D\ model\}$
$\mathcal{N}^{3D} = \{All\ nodes\ in\ 3D\ model\}$

Separate Interface Nodes:

$\mathcal{J}^{2D} = \{1_{2D}, 2_{2D}, 3_{2D}\}$
$\mathcal{J}^{3D} = \{1_{3D}, 2_{3D}, 3_{3D}, 4, \dots, 9\}$

Node Columns:

$\mathcal{C}(\mathcal{K} = 1_{2D}) = \{1_{3D}, 2_{3D}, 3_{3D}\}$
$\mathcal{C}(\mathcal{K} = 2_{2D}) = \{4, 5, 6\}$
$\mathcal{C}(\mathcal{K} = 3_{2D}) = \{7, 8, 9\}$



Coupling: Old system

Build the 'old' system of equations (19), as defined in (11)

$$
\begin{bmatrix}
\frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\
[0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\
\begin{bmatrix} \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \end{bmatrix} & [0] & \begin{bmatrix} \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{1}} & \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{2}} & \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{3}} \\ \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{1}} & \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{2}} & \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{3}} \\ \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{1}} & \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{2}} & \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{3}} \end{bmatrix} & [0] \\
[0] & \begin{bmatrix} \frac{\partial \boldsymbol{r}_{3D}^{1}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \vdots \\ \frac{\partial \boldsymbol{r}_{3D}^{9}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \end{bmatrix} & [0] & \begin{bmatrix} \frac{\partial \boldsymbol{r}_{3D}^{1}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \frac{\partial \boldsymbol{r}_{3D}^{1}}{\partial \boldsymbol{s}_{3D}^{9}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \boldsymbol{r}_{3D}^{9}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \frac{\partial \boldsymbol{r}_{3D}^{9}}{\partial \boldsymbol{s}_{3D}^{9}} \end{bmatrix}
\end{bmatrix}
\begin{Bmatrix} \Delta \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} \\ \Delta \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}} \end{Bmatrix}
= -\begin{Bmatrix} \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}(\boldsymbol{s}_{2D}) \\ \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}(\boldsymbol{s}_{3D}) \\ \begin{Bmatrix} \boldsymbol{r}_{2D}^{1}(\boldsymbol{s}_{2D}) \\ \boldsymbol{r}_{2D}^{2}(\boldsymbol{s}_{2D}) \\ \boldsymbol{r}_{2D}^{3}(\boldsymbol{s}_{2D}) \end{Bmatrix} \\ \begin{Bmatrix} \boldsymbol{r}_{3D}^{1}(\boldsymbol{s}_{3D}) \\ \vdots \\ \boldsymbol{r}_{3D}^{9}(\boldsymbol{s}_{3D}) \end{Bmatrix} \end{Bmatrix}
\qquad (19)
$$

Coupling: New residuals

Build the new residuals (20) using the old residuals and constraints, as per (12) and (13)

$$\boldsymbol{R}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} = \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}$$
$$\boldsymbol{R}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} = \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}$$
$$\boldsymbol{R}^{\mathcal{J}^{2D}} = \begin{Bmatrix} \boldsymbol{r}^1 = \boldsymbol{r}_{2D}^1 + \sum_{i\in\mathcal{C}(1)} \boldsymbol{r}_{3D}^i = \boldsymbol{r}_{2D}^1 + \boldsymbol{r}_{3D}^1 + \boldsymbol{r}_{3D}^2 + \boldsymbol{r}_{3D}^3 \\ \boldsymbol{r}^2 = \boldsymbol{r}_{2D}^2 + \sum_{i\in\mathcal{C}(2)} \boldsymbol{r}_{3D}^i = \boldsymbol{r}_{2D}^2 + \boldsymbol{r}_{3D}^4 + \boldsymbol{r}_{3D}^5 + \boldsymbol{r}_{3D}^6 \\ \boldsymbol{r}^3 = \boldsymbol{r}_{2D}^3 + \sum_{i\in\mathcal{C}(3)} \boldsymbol{r}_{3D}^i = \boldsymbol{r}_{2D}^3 + \boldsymbol{r}_{3D}^7 + \boldsymbol{r}_{3D}^8 + \boldsymbol{r}_{3D}^9 \end{Bmatrix}$$
$$\boldsymbol{R}^{\mathcal{J}^{3D}} = \boldsymbol{C} = \begin{Bmatrix} \boldsymbol{c}^1 = \boldsymbol{s}_{3D}^1 - \boldsymbol{s}_{2D}^1 \\ \boldsymbol{c}^2 = \boldsymbol{s}_{3D}^2 - \boldsymbol{s}_{2D}^1 \\ \boldsymbol{c}^3 = \boldsymbol{s}_{3D}^3 - \boldsymbol{s}_{2D}^1 \\ \boldsymbol{c}^4 = \boldsymbol{s}_{3D}^4 - \boldsymbol{s}_{2D}^2 \\ \boldsymbol{c}^5 = \boldsymbol{s}_{3D}^5 - \boldsymbol{s}_{2D}^2 \\ \boldsymbol{c}^6 = \boldsymbol{s}_{3D}^6 - \boldsymbol{s}_{2D}^2 \\ \boldsymbol{c}^7 = \boldsymbol{s}_{3D}^7 - \boldsymbol{s}_{2D}^3 \\ \boldsymbol{c}^8 = \boldsymbol{s}_{3D}^8 - \boldsymbol{s}_{2D}^3 \\ \boldsymbol{c}^9 = \boldsymbol{s}_{3D}^9 - \boldsymbol{s}_{2D}^3 \end{Bmatrix} \tag{20}$$

Coupling: New Jacobian

Build the modified Jacobian (21) for the Newton-Raphson iterations (9) using the modified residuals (20) and the derivatives (14) and (16)

$$(New\ Jacobian) = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{s}} = \begin{bmatrix} \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\ [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\ \begin{bmatrix} \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \end{bmatrix} & \begin{bmatrix} \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \frac{\partial \boldsymbol{r}^3}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \end{bmatrix} & \begin{bmatrix} \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{2D}^1} & \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{2D}^2} & \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{2D}^3} \\ \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{2D}^1} & \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{2D}^2} & \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{2D}^3} \\ \frac{\partial \boldsymbol{r}^3}{\partial \boldsymbol{s}_{2D}^1} & \frac{\partial \boldsymbol{r}^3}{\partial \boldsymbol{s}_{2D}^2} & \frac{\partial \boldsymbol{r}^3}{\partial \boldsymbol{s}_{2D}^3} \end{bmatrix} & \begin{bmatrix} \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{3D}^1} & \cdots & \frac{\partial \boldsymbol{r}^1}{\partial \boldsymbol{s}_{3D}^9} \\ \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{3D}^1} & \cdots & \frac{\partial \boldsymbol{r}^2}{\partial \boldsymbol{s}_{3D}^9} \\ \frac{\partial \boldsymbol{r}^3}{\partial \boldsymbol{s}_{3D}^1} & \cdots & \frac{\partial \boldsymbol{r}^3}{\partial \boldsymbol{s}_{3D}^9} \end{bmatrix} \\ [0] & [0] & \begin{bmatrix} \frac{\partial \boldsymbol{c}^1}{\partial \boldsymbol{s}_{2D}^1} & \frac{\partial \boldsymbol{c}^1}{\partial \boldsymbol{s}_{2D}^2} & \frac{\partial \boldsymbol{c}^1}{\partial \boldsymbol{s}_{2D}^3} \\ \vdots & \vdots & \vdots \\ \frac{\partial \boldsymbol{c}^9}{\partial \boldsymbol{s}_{2D}^1} & \frac{\partial \boldsymbol{c}^9}{\partial \boldsymbol{s}_{2D}^2} & \frac{\partial \boldsymbol{c}^9}{\partial \boldsymbol{s}_{2D}^3} \end{bmatrix} & \begin{bmatrix} \frac{\partial \boldsymbol{c}^1}{\partial \boldsymbol{s}_{3D}^1} & \cdots & \frac{\partial \boldsymbol{c}^1}{\partial \boldsymbol{s}_{3D}^9} \\ \vdots & \ddots & \vdots \\ \frac{\partial \boldsymbol{c}^9}{\partial \boldsymbol{s}_{3D}^1} & \cdots & \frac{\partial \boldsymbol{c}^9}{\partial \boldsymbol{s}_{3D}^9} \end{bmatrix} \end{bmatrix} \quad (21)$$

Coupling: New Jacobian

Expanding terms in the new Jacobian (21), making use of (15) and (17), we get (22)

$$
(New\ Jacobian) = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{s}} =
\begin{bmatrix}
\frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \frac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\
[0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \frac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\
\begin{bmatrix} \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \end{bmatrix} &
\begin{bmatrix} \sum_{i\in\mathcal{C}(1)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \sum_{i\in\mathcal{C}(2)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \sum_{i\in\mathcal{C}(3)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \end{bmatrix} &
\begin{bmatrix} \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{1}} & \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{2}} & \frac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{3}} \\ \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{1}} & \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{2}} & \frac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{3}} \\ \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{1}} & \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{2}} & \frac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{3}} \end{bmatrix} &
\begin{bmatrix} \sum_{i\in\mathcal{C}(1)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \sum_{i\in\mathcal{C}(1)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{9}} \\ \sum_{i\in\mathcal{C}(2)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \sum_{i\in\mathcal{C}(2)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{9}} \\ \sum_{i\in\mathcal{C}(3)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \sum_{i\in\mathcal{C}(3)} \frac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{9}} \end{bmatrix} \\
[0] & [0] &
\begin{bmatrix} -[I] & & [0] \\ -[I] & & \\ -[I] & & \\ & -[I] & \\ & -[I] & \\ & -[I] & \\ & & -[I] \\ [0] & & -[I] \\ & & -[I] \end{bmatrix} &
\begin{bmatrix} [I] & & & & & & & & \\ & [I] & & & & & & & \\ & & [I] & & & [0] & & & \\ & & & [I] & & & & & \\ & & & & [I] & & & & \\ & & & & & [I] & & & \\ & & [0] & & & & [I] & & \\ & & & & & & & [I] & \\ & & & & & & & & [I] \end{bmatrix}
\end{bmatrix} \quad (22)
$$

Coupling: New system

The Newton-Raphson iterations (9) for the coupled system are given by (23), where we have used the coupled Jacobian (22) and coupled residuals (20)

$$
\begin{bmatrix}
\dfrac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} & [0] & \dfrac{\partial \boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}}{\partial \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}}} & [0] \\
[0] & \dfrac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} & [0] & \dfrac{\partial \boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}}{\partial \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}}} \\
\begin{bmatrix} \dfrac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \dfrac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \\ \dfrac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}} \end{bmatrix} &
\begin{bmatrix} \sum_{i\in\mathcal{C}(1)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \sum_{i\in\mathcal{C}(2)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \\ \sum_{i\in\mathcal{C}(3)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}} \end{bmatrix} &
\begin{bmatrix} \dfrac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{1}} & \dfrac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{2}} & \dfrac{\partial \boldsymbol{r}_{2D}^{1}}{\partial \boldsymbol{s}_{2D}^{3}} \\ \dfrac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{1}} & \dfrac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{2}} & \dfrac{\partial \boldsymbol{r}_{2D}^{2}}{\partial \boldsymbol{s}_{2D}^{3}} \\ \dfrac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{1}} & \dfrac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{2}} & \dfrac{\partial \boldsymbol{r}_{2D}^{3}}{\partial \boldsymbol{s}_{2D}^{3}} \end{bmatrix} &
\begin{bmatrix} \sum_{i\in\mathcal{C}(1)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \sum_{i\in\mathcal{C}(1)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{9}} \\ \sum_{i\in\mathcal{C}(2)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \sum_{i\in\mathcal{C}(2)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{9}} \\ \sum_{i\in\mathcal{C}(3)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{1}} & \cdots & \sum_{i\in\mathcal{C}(3)} \dfrac{\partial \boldsymbol{r}_{3D}^{i}}{\partial \boldsymbol{s}_{3D}^{9}} \end{bmatrix} \\
[0] & [0] &
\begin{bmatrix} -[I] & & [0] \\ -[I] & & \\ -[I] & & \\ & -[I] & \\ & -[I] & \\ & -[I] & \\ & & -[I] \\ [0] & & -[I] \\ & & -[I] \end{bmatrix} &
\begin{bmatrix} [I] & & & & & & & & \\ & [I] & & & & & & & \\ & & [I] & & & & [0] & & \\ & & & [I] & & & & & \\ & & & & [I] & & & & \\ & & & & & [I] & & & \\ & & [0] & & & & [I] & & \\ & & & & & & & [I] & \\ & & & & & & & & [I] \end{bmatrix}
\end{bmatrix}
\begin{Bmatrix} \Delta \boldsymbol{s}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}} \\ \Delta \boldsymbol{s}_{2D}^{\mathcal{J}^{2D}} \\ \Delta \boldsymbol{s}_{3D}^{\mathcal{J}^{3D}} \end{Bmatrix}
= -
\begin{Bmatrix}
\boldsymbol{R}_{2D}^{\mathcal{N}^{2D}-\mathcal{J}^{2D}}(\boldsymbol{s}_{2D}) \\
\boldsymbol{R}_{3D}^{\mathcal{N}^{3D}-\mathcal{J}^{3D}}(\boldsymbol{s}_{3D}) \\
\begin{Bmatrix} \boldsymbol{r}_{2D}^{1}(\boldsymbol{s}_{2D}) + \boldsymbol{r}_{3D}^{1}(\boldsymbol{s}_{3D}) + \boldsymbol{r}_{3D}^{2}(\boldsymbol{s}_{3D}) + \boldsymbol{r}_{3D}^{3}(\boldsymbol{s}_{3D}) \\ \boldsymbol{r}_{2D}^{2}(\boldsymbol{s}_{2D}) + \boldsymbol{r}_{3D}^{4}(\boldsymbol{s}_{3D}) + \boldsymbol{r}_{3D}^{5}(\boldsymbol{s}_{3D}) + \boldsymbol{r}_{3D}^{6}(\boldsymbol{s}_{3D}) \\ \boldsymbol{r}_{2D}^{3}(\boldsymbol{s}_{2D}) + \boldsymbol{r}_{3D}^{7}(\boldsymbol{s}_{3D}) + \boldsymbol{r}_{3D}^{8}(\boldsymbol{s}_{3D}) + \boldsymbol{r}_{3D}^{9}(\boldsymbol{s}_{3D}) \end{Bmatrix} \\
\begin{Bmatrix} \boldsymbol{s}_{3D}^{1} - \boldsymbol{s}_{2D}^{1} \\ \boldsymbol{s}_{3D}^{2} - \boldsymbol{s}_{2D}^{1} \\ \boldsymbol{s}_{3D}^{3} - \boldsymbol{s}_{2D}^{1} \\ \boldsymbol{s}_{3D}^{4} - \boldsymbol{s}_{2D}^{2} \\ \boldsymbol{s}_{3D}^{5} - \boldsymbol{s}_{2D}^{2} \\ \boldsymbol{s}_{3D}^{6} - \boldsymbol{s}_{2D}^{2} \\ \boldsymbol{s}_{3D}^{7} - \boldsymbol{s}_{2D}^{3} \\ \boldsymbol{s}_{3D}^{8} - \boldsymbol{s}_{2D}^{3} \\ \boldsymbol{s}_{3D}^{9} - \boldsymbol{s}_{2D}^{3} \end{Bmatrix}
\end{Bmatrix}
\tag{23}
$$

Discussion

Conservation of quantities at the 2D-3D interface

- Enforced continuity of water surface elevation
- Enforced continuity of depth averaged velocity

Solvability of the coupled 2D-3D system

- Started with the individual (non-coupled) models, solvable upon application of boundary conditions at the interface
- New 2D interface residuals obtained by summing up linearly independent interface residuals
- New 3D interface residuals set to be linearly independent constraints

Bathymetry fixed in time

- Not applicable for sediment transport, for example

Transport of constituents in the coupled 2D-3D model

- Nearly identical treatment, 1 equation per node, per constituent, instead of 3 equations per node for SW

Continuity equation and different solution variables need separate treatment

Acknowledgments

This project was funded by the US Army Corps of Engineers, project PP-EQM-KY07-001-P3, under the Department of Defense (DOD) High Performance Computing Modernization Program (HPCMP), contract No. GS04T09DBC0017.

We acknowledge the following people for their contributions to this work:

- Dr. Clint Dawson (Professor, The University of Texas at Austin)
- Dr. Corey Trahan (Research Physicist, US Army Corps of Engineers)
- Dr. Matthew Farthing (Research Hydraulic Engineer, US Army Corps of Engineers)
- Dr. Charlie Berger (Research Hydraulic Engineer, US Army Corps of Engineers)

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Questions?

Thank You!