High-resolution modeling of three-dimensional circulation in the Gulf of Mexico

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Outline

3D Shallow Water Equations
Accurate computation of baroclinic pressure gradients and horizontal diffusion terms
Numerical experiments –toy problems
Bathymetry smoothing
Gaussian filter for bathymetry/velocity
Biharmonic viscosity operator
Smagorinsky/Leith viscosity parameter
Simulation results of Gulf of Mexico

Storm surge during Hurricane Katrina.
3D Shallow Water Equations (SWE)

Spherical coordinate:

- Continuity:

\[
\frac{1}{R \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{1}{R \cos \phi} \frac{\partial (\nu \cos \phi)}{\partial \phi} + \frac{\partial w}{\partial z} + \frac{2w}{R} = 0.
\]

- Horizontal momentum:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{v}{R} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} &= f v - \frac{1}{R \cos \phi} \frac{\partial [g(\zeta - \alpha \eta) + p_s/\rho_0]}{\partial \lambda} + \frac{\partial}{\partial z}\left(\frac{T_{z\lambda}}{\rho_0}\right) - b_\lambda + m_\lambda, \\
\frac{\partial u}{\partial t} + \frac{u}{R \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{v}{R} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} &= -fu - \frac{1}{R} \frac{\partial [g(\zeta - \alpha \eta) + p_s/\rho_0]}{\partial \phi} + \frac{\partial}{\partial z}\left(\frac{T_{z\phi}}{\rho_0}\right) - b_\phi + m_\phi,
\end{align*}
\]

where:

\[
b_\lambda = \frac{g}{\rho_0 R \cos \phi} \frac{\partial}{\partial \lambda} \int_z^\zeta \rho \, dz, \quad b_\phi = \frac{g}{\rho_0 R \cos \phi} \frac{\partial}{\partial \phi} \int_z^\zeta \rho \, dz.
\]

ADCIRC does not solve the primitive continuity equation directly. It solves the generalized wave continuity equation, which is a combination of the primitive continuity equation, its temporal derivative, and spatial derivatives of the depth-integrated momentum equations.
3D Shallow Water Equations (SWE)

Cartesian coordinate:

\[
S_p \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]

\[
\frac{\partial u}{\partial t} + uS_p \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv = -S_p \frac{\partial [g(\zeta - \alpha \eta) + p_s/\rho_0]}{\partial x} + \frac{\partial}{\partial z} \left( \frac{\tau_{zx}}{\rho_0} \right) - b_x + m_x,
\]

\[
\frac{\partial v}{\partial t} + uS_p \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu = -\frac{\partial [g(\zeta - \alpha \eta) + p_s/\rho_0]}{\partial y} + \frac{\partial}{\partial z} \left( \frac{\tau_{zy}}{\rho_0} \right) - b_y + m_y,
\]

where \( S_p = \frac{\cos \phi_0}{\cos \phi} \)

Baroclinic pressure gradient:

\[
b_x = \frac{g}{\rho_0} S_p \frac{\partial}{\partial x} \int_z^\zeta \rho \, dz = \frac{g}{\rho_0} S_p \frac{\partial}{\partial x} \int_z^\zeta (\rho - \rho_o) \, dz,
\]

\[
b_y = \frac{g}{\rho_0} \frac{\partial}{\partial y} \int_z^\zeta \rho \, dz = \frac{g}{\rho_0} \frac{\partial}{\partial y} \int_z^\zeta (\rho - \rho_o) \, dz.
\]
Sigma coordinate:

\[ x_\sigma = x, \]
\[ y_\sigma = y, \]
\[ \sigma = a + \frac{a-b}{H}(z - \zeta), \quad a = +1, \quad b = -1, \]
\[ t_\sigma = t. \]
Sigma coordinate:

\[
\begin{align*}
\frac{\partial u}{\partial t_\sigma} + uS_p \frac{\partial u}{\partial x_\sigma} + v \frac{\partial u}{\partial y_\sigma} + w_\sigma \left( \frac{a - b}{H} \right) \frac{\partial u}{\partial \sigma} - f v &= \\
- S_p \frac{\partial [g(\zeta - \alpha \eta) + p_s/\rho_0]}{\partial x} + \left( \frac{a - b}{H} \right) \frac{\partial}{\partial \sigma} \left( \frac{\tau_{zx}}{\rho_0} \right) - b_x + m_x, \\
\frac{\partial v}{\partial t_\sigma} + uS_p \frac{\partial v}{\partial x_\sigma} + v \frac{\partial v}{\partial y_\sigma} + w_\sigma \left( \frac{a - b}{H} \right) \frac{\partial v}{\partial \sigma} + f u &= \\
- \frac{\partial [g(\zeta - \alpha \eta) + p_s/\rho_0]}{\partial y} + \left( \frac{a - b}{H} \right) \frac{\partial}{\partial \sigma} \left( \frac{\tau_{zy}}{\rho_0} \right) - b_y + m_y, \\
w_\sigma = w - \left( \frac{\sigma - b}{a - b} \right) \frac{\partial \zeta}{\partial x} - u \left[ \left( \frac{\sigma - b}{a - b} \right) \frac{\partial \zeta}{\partial x} + \left( \frac{\sigma - a}{a - b} \right) \frac{\partial h}{\partial x} \right] - v \left[ \left( \frac{\sigma - b}{a - b} \right) \frac{\partial \zeta}{\partial y} + \left( \frac{\sigma - a}{a - b} \right) \frac{\partial h}{\partial y} \right].
\end{align*}
\]
Baroclinic pressure gradient (BPG)

Sigma coordinate:

\[ P = \frac{g}{\rho_0} \int_{z}^{\zeta} \rho \, dz \]

\[ b_x = \frac{\partial P}{\partial x} = \frac{\partial P}{\partial x_\sigma} + \frac{\partial P}{\partial \sigma} \frac{\partial \sigma}{\partial x} \]
Baroclinic pressure gradient: z-coordinate

Scheme 1:

a) Evaluate pressure at the vertices of the projected elements.
b) Compute constant $BPG_{x/y}$ for each element.
c) Compute the weak form: $(BPG_{x/y}, \text{test function})$. 
Baroclinic pressure gradient: z-coordinate

Scheme 2:

a) Evaluate pressure at the vertices of the projected elements.
b) Compute constant BPG$_{x/y}$ for each element.
c) Compute BPG$_{x/y}$ for the vertex of interest via SPR.

Baroclinic pressure gradient: z-coordinate

**Scheme 2:**

a) Evaluate pressure at the vertices of the projected elements.
b) Compute constant $BPG_{x/y}$ for each element.
c) Compute $BPG_{x/y}$ for the vertex of interest via SPR.
d) Repeat steps a-c for all vertices.
Baroclinic pressure gradient: z-coordinate

Scheme 2:

a) Evaluate pressure at the vertices of the projected elements.
b) Compute constant $BPG_{x/y}$ for each element.
c) Compute $BPG_{x/y}$ for the vertex of interest via SPR.
d) Repeat steps a-c for all vertices.
e) Evaluate $BPG_{x/y}$ at the vertices of the projected elements.
f) Compute the weak form: $(BPG_{x/y}, \text{test function})$, where now $BPG_{x/y}$ varies linearly within an element, and is continuous across elements.
Numerical experiment: BPG error for stratified fluid

Scheme 1: linear vertical interpolation

Scheme 1: cubic vertical interpolation

Scheme 2: cubic vertical interpolation

$\rho_{\text{top}} = 34$ psu
$\rho_{\text{bot}} = 35$ psu
21 sigma layers
$rx_0 = 0.45$
$rx_1 = 17.6$
Numerical experiment: lock-exchange problem

Scheme 1: cubic vertical interpolation

$\rho_{\text{left}} = 35 \text{ psu}$
$\rho_{\text{right}} = 34 \text{ psu}$
$\sigma_{x_0} = 0.18$
$\sigma_{x_1} = 7.1$

Scheme 2: cubic vertical interpolation
Transport of salinity and temperature – horizontal diffusion

\[ \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} - \frac{\partial}{\partial x} \left( N_h \frac{\partial c}{\partial x} \right) - \frac{\partial}{\partial y} \left( N_h \frac{\partial c}{\partial y} \right) - \frac{\partial}{\partial z} \left( N_v \frac{\partial c}{\partial z} \right) = 0 \]

In sigma coordinate:

\[ \frac{\partial}{\partial x} \left( N_h \frac{\partial c}{\partial x} \right) = \frac{\partial}{\partial x_\sigma} \left( N_h \frac{\partial c}{\partial x_\sigma} \right) + \frac{\partial}{\partial x_\sigma} \left( N_h \frac{\partial c}{\partial \sigma} \frac{\partial \sigma}{\partial x} \right) + \frac{\partial}{\partial \sigma} \left( N_h \frac{\partial c}{\partial x_\sigma} \frac{\partial \sigma}{\partial x} \right) + \frac{\partial}{\partial \sigma} \left( N_h \frac{\partial c}{\partial \sigma} \frac{\partial \sigma}{\partial x} \right) \frac{\partial \sigma}{\partial x} . \]

Iso-sigma diffusion:

\[ \frac{\partial}{\partial x} \left( N_h \frac{\partial c}{\partial x} \right) \approx \frac{\partial}{\partial x_\sigma} \left( N_h \frac{\partial c}{\partial x_\sigma} \right) . \]
Transport of salinity and temperature – horizontal diffusion

Sigma+Cartesian coordinate:

\[
\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x_\sigma} + v \frac{\partial c}{\partial y_\sigma} + w_\sigma \left( \frac{a-b}{H} \right) \frac{\partial c}{\partial \sigma} - \frac{\partial}{\partial x} \left( N_h \frac{\partial c}{\partial x} \right) - \frac{\partial}{\partial y} \left( N_h \frac{\partial c}{\partial y} \right) - \left( \frac{a-b}{H} \right)^2 \frac{\partial}{\partial \sigma} \left( N_v \frac{\partial c}{\partial \sigma} \right) = 0.
\]

Iso-sigma diffusion (lock-exchange after 40 days).

Correct treatment of horizontal diffusion (lock-exchange after 40 days).
Numerical experiment: lock-exchange problem

-t = 0
-t = 1 hr
-t = 6 hr
-t = 1 day
-t = 4 days
-t = 10 days
-t = 40 days
Steep bathymetry - smoothing

\[ rx_0 = 0.45, \text{ } rx_1 = 17.6, \text{ maximum element slope} = 0.31 \]
Unstable (<1 day)

\[ rx_0 = 0.09, \text{ } rx_1 = 3.6, \text{ maximum element slope} = 0.20 \]
Stable (180 days)
Steep bathymetry - smoothing for rx0

Bathymetry roughness indicators:

\[ r_{x_0} = \frac{|h_i - h_j|}{h_i + h_j}, \quad \text{(range: 0.15 - 0.2)} \]

\[ r_{x_1} = \frac{|h_i^k - h_j^k + h_i^{k-1} - h_j^{k-1}|}{h_i^k + h_j^k - h_i^{k-1} - h_j^{k-1}}, \quad \text{(range: 3 - 8)} \]

Algorithm 1 Smoothing bathymetry for \( r_{x_0} \).

1: \textbf{for} iter = 1 : iter_{max} \textbf{do}
2: \hspace{1em} \textbf{loop} over all elements
3: \hspace{2em} \textbf{loop} over all edges of an element
4: \hspace{3em} \textbf{if} \ \frac{|h_i - h_j|}{h_i + h_j} > r_{desired} \textbf{then}
5: \hspace{4em} h_i = h_i - \delta \text{ and } h_j = h_j + \delta \text{ s.t. } \frac{|h_i - h_j|}{h_i + h_j} = r_{desired}
6: \hspace{3em} \textbf{end if}
7: \hspace{2em} \textbf{end loop}
8: \hspace{1em} \textbf{end loop}
9: \textbf{end for}

Algorithm 1 Smoothing bathymetry for $rx_1$.

1: for $\text{iter} = 1 : \text{iter}_{\text{max}}$ do
2:     loop over all elements
3:         loop over all edges of an element
4:             loop over all vertical layers
5:                 if $\frac{|h_i^k-h_j^k+h_{k-1}^k-h_{k-1}^k|}{h_i^k+h_j^k-h_{k-1}^k-h_{k-1}^k} > r_{\text{desired}}$ then
6:                     $h_i = h_i - \delta$ and $h_j = h_j + \delta$
7:                 else
8:                     s.t. $\frac{|h_i^k-h_j^k+h_{k-1}^k-h_{k-1}^k|}{h_i^k+h_j^k-h_{k-1}^k-h_{k-1}^k} > r_{\text{desired}}$
9:         end loop
10:     end loop
11: end loop
12: end for
Algorithm 3 Gaussian smoother: $q \rightarrow q_{\text{smooth}}$

1: loop over all nodes  
2: \hspace{1cm} loop over all neighbor nodes of the above node \hspace{1cm} $\triangleright$ Number of neighbors $= N$  
3: \hspace{1cm} find $l_{\text{max}}$: maximum distance from the center node to its neighbors  
4: \hspace{1cm} $\sigma = l_{\text{max}} \times \alpha$ \hspace{1cm} $\triangleright$ e.g. $\alpha = 0.7$  
5: \hspace{1cm} end loop  
6: loop over the center node and all of its neighbors  
7: \hspace{1cm} find the distance $l_i$ from the central node to the neighbor node  
8: compute node's weight: $w_i = \exp\left(-\frac{l_i^2}{2\sigma^2}\right)$  
9: normalize weights, such that: $\Sigma_0^N w_i = 1$  
10: $q_{\text{smooth}}(0) = \Sigma_0^N q_i w_i$  
11: end loop  
12: end loop

Another bathymetry smoothing technique: diffusion (MITgcm)
SL-16 finite element mesh for Southeastern Louisiana

4'751'299 elements
2'437'172 nodes
SL-16 after smoothing for rx0 and rx1
SL-16: rx0, rx1 + repeated application of the Gaussian filter
Slope parameter
Biharmonic viscosity operator

- Laplacian viscosity operator:

\[ m_x = k_L \nabla^2 u \]

- Biharmonic viscosity operator (enhanced scale-selectivity):

\[ m_x = -k_B \nabla^2 \nabla^2 u \]

- One-dimensional example:

\[ u_t = k_L u_{xx} \]

\[ u_t = -k_B u_{xxxx} \]
Biharmonic viscosity operator – overshoot/undershoot

\[
\begin{align*}
\frac{\partial C}{\partial t} &= -\frac{\partial^4 C}{\partial x^4} \\
C(0, x) &= C_0(x)
\end{align*}
\]

\[C_0(x) = \begin{cases} 
0 & \text{for } |x| > 1 \\
1 & \text{for } |x| < 1 
\end{cases}\]

Smagorinsky viscosity

3D turbulence theory - harmonic operator:

\[ A_{Smag} = \left( \frac{L}{\pi} \right)^2 L^2 |D|, \]
\[ |D| = \sqrt{\left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2}, \]

3D turbulence theory - biharmonic operator:

\[ A_{Smag} = \left( \frac{L}{\pi} \right)^2 \frac{L^4}{8} |D|, \]
\[ |D| = \sqrt{\left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2}, \]
Leith viscosity

2D turbulence theory - harmonic operator:

\[ A_{Leith} = \left( \frac{\Lambda}{\pi} \right)^3 L^3 |\nabla \omega| \]

\[ |\nabla \omega| = \sqrt{\left( \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right)^2 + \left( \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right)^2} \]

2D turbulence theory - biharmonic operator:

\[ A_{Leith} = \left( \frac{\Lambda}{\pi} \right)^3 \frac{L^5}{8} |\nabla \omega| \]

\[ |\nabla \omega| = \sqrt{\left( \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right)^2 + \left( \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right)^2} \]

CFL condition:

\[ A_{Laplacian} \lessdot \frac{L^2}{4 \Delta t} \]

\[ A_{Biharmonic} \lessdot \frac{L^4}{32 \Delta t} \]
Modified Leith viscosity

2D turbulence theory - biharmonic operator:

\[ A_{Leith} = \left( \frac{1}{\pi} \right)^3 \frac{L^5}{8} \sqrt{\Lambda^6 |\nabla \omega|^2 + \Lambda_d^6 |\nabla \nabla \cdot u_h|^2} \]

\[ |\nabla \omega| = \sqrt{\left[ \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right]^2 + \left[ \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right]^2} \]

\[ |\nabla \nabla \cdot u_h| = \sqrt{\left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2 + \left[ \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2} \]
Modified Leith viscosity – consequences: less noise

Vertical velocity from a simulation of spindown and instability of a temperature front in a reentrant channel. A simulation with the Leith viscosity applied to the horizontal velocities (left) and a simulation with the modified Leith viscosity (right) are shown. Light colors are near zero; colors represent upward or downward motion.

Modified Leith viscosity – consequences: better stability

A Global Ocean Simulation with MITgcm:

Maximum Courant number. *Gray line is from the Leith integration, and black line is from the modified Leith integration.*

Velocity field during a spin-up simulation on the SL16 mesh with wind and baroclinic forcings (June 2010). The Loop Current and other eddies can be seen in the figure.
Gulf of Mexico simulation results: full-run

Velocity field during a prognostic simulation on the SL16 mesh (June 16th 2010). The Loop Current and other eddies can be seen in the figure.
Gulf of Mexico simulation results: temperature

Temperature field during a prognostic simulation on the SL16 mesh (June 16th 2010).
Gulf of Mexico simulation results: salinity

Salinity field during a prognostic simulation on the SL16 mesh (June 16th 2010).
Concluding Remarks

- Algorithmic improvements, leading to the accurate computation of the baroclinic pressure gradients and horizontal diffusion terms of the transport equations, by evaluating these terms directly in the z-coordinate, and using higher-order vertical interpolation.

- Systematic schemes for smoothing bathymetry.

- Better accounting for horizontal viscosity by using a biharmonic operator, which is more scale-selective, and also computing the viscosity parameter at every grid point, dynamically, according to a modified version of Leith.

- Stable simulation of the entire Gulf of Mexico on a high-resolution mesh during 2010/06/13 - 2010/06/25.
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